Ergodic Theory and Measured Group Theory
Lecture 4

Examples of ergodic transformations.

- Irrational rotation. Let $\alpha \in [-\pi, \pi)$ be s.t. $\frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$.
  Let $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong S^1$ and let $T_\alpha : \mathbb{T} \to \mathbb{T}$ be the rotation by α.

99% Lemma (special case of Lebesgue diff. theorem). For any positively-measured subset $A \subseteq [0,1)$, there is an open interval $I \subseteq [0,1)$ s.t. $\geq 99\%$ of I is occupied by A, i.e. $\frac{\lambda(A \cap I)}{\lambda(I)} \geq 0.99$.

Proof. Fix $\varepsilon > 0$. $\exists$ an open $U \subseteq [0,1)$ s.t. $U \supseteq A$ s.t. $\lambda(A)/\lambda(U) > 1-\varepsilon$ (take U s.t. $\lambda(U \setminus A) < \varepsilon \cdot \lambda(A)$ so $\frac{\lambda(A)}{\lambda(U)} = \frac{\lambda(A)}{\lambda(A) + \lambda(U \setminus A)} > \frac{\lambda(A)}{\lambda(A) + \lambda(A)\varepsilon}$ $= \frac{1}{1+\varepsilon} \geq 1-\varepsilon$).

But $U = \bigsqcup_{n \in \mathbb{N}} I_n$ disjoint union of open intervals and

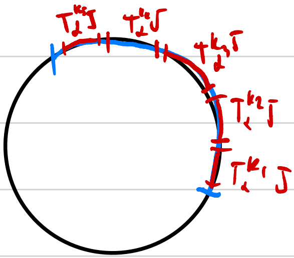
$$\frac{\lambda(A)}{\lambda(U)} = \text{convex combination of } \frac{\lambda(A \cap I_n)}{\lambda(I_n)}.$$

Indeed, $\frac{\lambda(A)}{\lambda(U)} = \frac{1}{\lambda(U)} \cdot \sum_n \lambda(A \cap I_n) = \sum_n \frac{\lambda(I_n)}{\lambda(U)} \cdot \frac{\lambda(A \cap I_n)}{\lambda(I_n)}$.

Hence, at least for one n, $\frac{\lambda(A \cap I_n)}{\lambda(I_n)} > 1-\varepsilon$ because o.w. $\frac{\lambda(A)}{\lambda(U)} \leq$ convex combo of $1-\varepsilon = 1-\varepsilon$, contradiction. $\square$

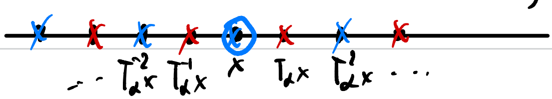
Prop. Irrational rotation T_α is ergodic.

Proof. Towards a contradiction, let A be a T_α-invariant measurable set s.t. both A and A^c have positive measure.
By the 99% lemma, $\exists$ intervals (i.e. segments in S^1) I and J s.t. 99% of I is A and 99% of J is A^c.
Say $|J| \le |I|$. Invariance of A^c implies that the translates $T^k J$ of J are still 99% A^c. We cover at least half of I by translates of J, so $0.99 \cdot \frac{1}{2}$ of I is A^c but it's > 1% of I, a contradiction. How do we cover?



By the density of the orbit of the left endpoint of J, we can cover $\left\lfloor \frac{|I|}{|J|} \right\rfloor \cdot |J| / |I|$ fraction of I. $\square$

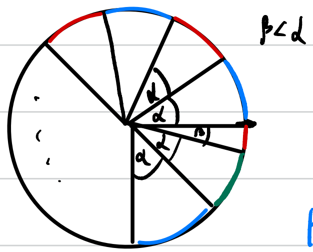
Application to graph coloring. Recall G_{T_α} is the graph of T_α, consider its undirected version. Then every connected component is a $\mathbb{Z}$-line



Thus, this graph is 2-colorable using Axiom of Choice to pick a starting point from each of the continuum-many components.

But what if we want a measurable coloring, i.e. each color is a measurable set? How many colors do we need?

We can do it with 3 colors:



However, 2 measurable colors aren't enough:

Cor. For an irrational rotation T_α, G_{T_α} is not measurably 2-colorable.

Proof. Suppose that it is, so $\exists$ set (i.e. a color) A s.t. $T_\alpha A = A^c$. Because T_α is measure-preserving, A and A^c have to have the same measure, hence $\lambda(A) = \lambda(A^c) = \frac{1}{2}$.
But $T_\alpha^2 A = A$ and $T_\alpha^2 A^c = A^c$, and $T_\alpha^2 = T_{2\alpha}$, so A is a $T_{2\alpha}$-invariant A. But $T_{2\alpha}$ is still an irrational rotation, hence ergodic. Thus, $\lambda(A) = 0$ or 1, a contradiction. $\square$

- <u>One sided shift</u>. $s: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with any Bernoulli measure $\mu := \nu^{\mathbb{N}}$, where ν is a measure on $2 := \{0,1\}$. We'll prove a stronger statement than ergodicity, namely, mixing.

Def. A pmp transformation $T: (X,\mu) \to (X,\mu)$ is called *mixing* if for any meas. $A, B \subseteq X$, $\mu(A \cap T^{-n}B) \to \mu(A) \cdot \overset{\mu(T^{-n}B)}{\overset{\text{Prop II}}{\mu(B)}}$ as $n \to \infty$.

Probability detour. Sets A, B are called *independent* if $\mu(A \cap B) = \mu(A)\mu(B)$, i.e. $\frac{\mu(B)}{\mu(X)} = \frac{\mu(B \cap A)}{\mu(A)}$.

So mixing says that eventually A and $T^{-n}B$ become almost independent.

Mixing $\Rightarrow$ ergodic. Let A be a T-invariant, so $T^{-n}A = A$. Take $B := A$.

Then $\mu(A \cap T^{-n}A) \to \mu(A)^2$ as $n \to \infty$

$\qquad \mu(A \overset{\shortparallel}{\cap} A) = \mu(A)$.

Thus, $\mu(A)^2 = \mu(A)$ so $\mu(A) \in \{0, 1\}$. $\square$

Prop. The shift S is mixing.

Proof. First we prove this in case A, B are basic clopen sets, i.e. $A = [s]$, $B = [t]$, where $s, t \in 2^{<\mathbb{N}}$.

(Define $[s] := \{x \in 2^{\mathbb{N}} : x \text{ starts with } s\}$.)

$[s]$: $\underbrace{0\,|\,1\,|\,1\,|\,0}_{s}\,|\,*\,|\,*\,|\,*\,|\,*\,|$

$[t]$: $\underbrace{1\,|\,1\,|\,0\,|\,1\,|\,0}_{t}\,|\,*\,|\,*\,|\,*\,|\,*\,|\,*\,|\,*\,|$

Let $n \geq \text{length}(s)$. $T^{-n}[t] = \overbrace{\boxed{*}\boxed{*}\boxed{*}\boxed{*}}^{|s|}\boxed{*}\boxed{*}\overbrace{\boxed{1}\boxed{1}\boxed{0}\boxed{1}\boxed{0}}^{t}\boxed{*}\boxed{*}\boxed{*}\cdots$

$[s] = \boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{*}\boxed{*}\boxed{*}\boxed{*}\boxed{*}\boxed{*}\cdots$

$[s] \cap T^{-n}[t] := \underbrace{\boxed{0}\boxed{1}\boxed{1}\boxed{1}\boxed{0}}_{s}\boxed{*}\underbrace{\boxed{*}\boxed{1}\boxed{1}\boxed{0}\boxed{1}\boxed{0}}_{t}\boxed{*}\boxed{*}\boxed{*}\cdots$

$\mu([s] \cap T^{-n}[t]) = \mu([s]) \cdot \mu([t])$. $\square$ (for basic clopen)

For general A, B, approximate A and B by finite disjoint unions of basic clopen sets (exercise). $\square$